Set of Permutations of the Natural Numbers

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Abstract

Almost all mathematicians are familiar with permutations of a finite set of elements. But, we can also extend this idea of permutation to infinite sets. This paper aims to investigate the set of permutations of the natural numbers.

General Definition

Each permutation can be described as a bijective function f which maps elements from a set A to the same set A.

$$\forall x \in A \; \exists_1 y \in A : f(x) = y$$

In this paper we will also define the natural numbers $\mathbb{N}$ to include 0.

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}$$

Finally, let $\mathfrak{p}$ be the set of permutations of $\mathbb{N}$.

An Attempt to Count $\mathfrak{p}$

We will attempt to construct $\mathfrak{p}$ in a way that demonstrates it to be countable. Our constructed set, we will call $\mathfrak{t}$.

Since $\mathfrak{t}$ is countable it can be represented as a sequence of permutations (f_i). Let f_0 map the natural numbers in order from least to greatest.

$$f_0(0) = 0, f_0(1) = 1, f_0(2) = 2, \dots$$

Now let $F_n = \{f_i : i < n!\}$, such that F_n contains all permutations of f_0 while remapping only domain values $x < n$. For example,

$$F_3 = \left\{ \begin{array}{llllll} f_0 & \to & 1 & 2 & 3 & 4 & \dots \\ f_1 & \to & 2 & 1 & 3 & 4 & \dots \\ f_2 & \to & 1 & 3 & 2 & 4 & \dots \\ f_3 & \to & 2 & 3 & 1 & 4 & \dots \\ f_4 & \to & 3 & 1 & 2 & 4 & \dots \\ f_5 & \to & 3 & 2 & 1 & 4 & \dots \end{array} \right\}$$

A consequence of this is that $F_n \subset F_{n+1}$. $\mathfrak{t}$ should now be definable as the union of F_n for all natural numbers n.

$$\mathfrak{t} = \bigcup_{n \in \mathbb{N}} F_n$$

This would also give us the cardinality of $\mathfrak{t}$,

$$\begin{aligned} |\mathfrak{t}| &= \left| \bigcup_{n \in \mathbb{N}} F_n \right| \\ &\leq \sum_{n \in \mathbb{N}} |F_n| \\ &\leq |\mathbb{N}| \cdot \aleph_0 \\ &= \aleph_0^2 \\ &= \aleph_0 \end{aligned}$$

Since $\mathfrak{t}$ is infinite, $|\mathfrak{t}| = \aleph_0$.

Based off of our construction of $\mathfrak{t}$, we can determine that for all $f_i \in \mathfrak{t}$, there exists a natural number N such that for all $c > N$, $f_i(c) = f_0(c)$.

$$\forall f_i \in \mathfrak{t} \;\; \exists N \in \mathbb{N} : \forall c > N \;\; f_i(c) = f_0(c)$$

In other words, each permutation in $\mathfrak{t}$ eventually becomes equivalent to f_0 for domain values greater than some N.

Consider the permutation f^* which flips the positions of $2x$ and $2x+1$ for all natural numbers x using f_0 as a base.

$$(f^*(i)) = (1, 0, 3, 2, 5, 4, 7, 6, \dots)$$

It is clearly evident that there does not exist a natural number N for which domain values $c > N$ imply $f^*(c) = f_0(c)$.

$$\nexists N \in \mathbb{N} : \forall c > N \;\; f^*(c) = f_0(c)$$

Then $f^* \notin \mathfrak{t}$, even though it is a permutation of the natural numbers. Therefore, our constructed set $\mathfrak{t}$ is not equal to the set of permutations of the natural numbers $\mathfrak{p}$. Although, it is true that $\mathfrak{t} \subset \mathfrak{p}$.

Cardinality of $\mathfrak{p}$

Upon first inspection, I used rules for finite permutations and extended them to make conclusions about infinite permutations. The first rule is that the number of permutations of n many items is n factorial. From this, we propose that the cardinality of $\mathfrak{p}$ is equal to $\aleph_0$ factorial.

$$|\mathfrak{p}| = \aleph_0!$$

Assuming the continuum hypothesis and using theories about the sizes of different forms for sufficiently large x, specifically,

$$x! > \alpha^x$$

We conclude that $\aleph_0$ factorial is equal $\aleph_1$.

$$\aleph_0! = 2^{\aleph_0} = \aleph_1$$

We will now prove that the $\mathfrak{p}$ is uncountably infinite using the definition of uncountable sets.

$$|\mathfrak{p}| = \aleph_1$$

Proof. Suppose that $\mathfrak{p}$ is countable. We can now order $\mathfrak{p}$ by bijectively mapping each permutation $f_n \in \mathfrak{p}$ to the corresponding natural number n such that,

$$\mathfrak{p} = \{f_n : n \in \mathbb{N}\}$$

Construct a permutation $g \in \mathfrak{p}$ such that,

$$g(2x) = \begin{cases} 2x & f_x(2x) > f_x(2x+1) \\ 2x+1 & \text{else} \end{cases}$$

and,

$$g(2x+1) = \begin{cases} 2x+1 & f_x(2x) > f_x(2x+1) \\ 2x & \text{else} \end{cases}$$

The bijectivity of g can be proven through proof by induction, but in the interest of saving time, we will skip doing so here.

Since $\mathfrak{p}$ is countable, there must exist a natural number N such that $g = f_N$. By the definition of g,

$$g(2N) = \begin{cases} 2N & g(2N) > g(2N+1) \\ 2N+1 & \text{else} \end{cases}$$

$$g(2N+1) = \begin{cases} 2N+1 & g(2N) > g(2N+1) \\ 2N & \text{else} \end{cases}$$

This leaves us with two cases:

$$\begin{array}{ll} \text{Case 1:} & g(2N) > g(2N+1) \\ \text{Case 2:} & g(2N) < g(2N+1) \end{array}$$

Since g is a bijective mapping, every value will be unique. Hence, $g(2N)$ will never equal $g(2N+1)$, meaning case 2 does not need to account for it.

In *case 1*,

$$g(2N) = 2N$$
$$g(2N+1) = 2N+1$$

Therefore $g(2N) < g(2N+1)$, but this contradicts our assumption that $g(2N) > g(2N+1)$.

In *case 2*,

$$g(2N) = 2N+1$$
$$g(2N+1) = 2N$$

Therefore $g(2N) > g(2N+1)$, but this contradicts our assumption that $g(2N) < g(2N+1)$.

Since both cases produce contradictions, our original assumption that $\mathfrak{p}$ was countably infinite is false. Therefore, $\mathfrak{p}$ is uncountably infinite through proof by contradiction.

$$|\mathfrak{p}| = \aleph_1$$

□

Countably Infinite Set of Permutations

Based off of our previous proof, it is trivial to see that the set of permutations of *any* countably infinite set is itself uncountably infinite.

There is no infinite cardinal number between $\aleph_0$ and a finite number. Knowing also that the set of permutations of a finite set is itself finite, we conclude that there does not exists a set of permutations $\mathfrak{q}$ which is countably infinite, has a cardinality of $\aleph_0$.

$$\nexists \mathfrak{q} : |\mathfrak{q}| = \aleph_0$$

A set of permutations must be either finite or uncountably infinite.

$$\forall \mathfrak{q} (|\mathfrak{q}| \in \mathbb{N} \vee |\mathfrak{q}| = \aleph_1)$$